It is clear that the systematic investigation of all such determinants will disclose the lowest values of m and n which will serve for a given power series, and that we may then actually determine the irreducible algebraic equation which the given power-series will actually satisfy. If no finite values of m and n can be found which satisfy the conditions of the theorem, the given power-series defines a transcendental function.

The remarkable theorem of EISENSTEIN,[1] which furnishes a necessary condition on the coefficients a_k in order that w may be algebraic, in the special case when the coefficients a_k are all rational numbers, may be regarded as a consequence of the general theorems stated in this paper.

CONCERNING A METHOD FOR FINDING A PARTICULAR INTEGRAL.[2]

By ARTHUR B. COBLE, University of Illinois.

The problem considered here is the determination of a particular integral U of the linear differential equation with constant coefficients,

$$f(D)y \equiv (k_0D^n + k_1D^{n-1} + \cdots + k_n)y = X\left(D = \frac{d}{dx}\right), \tag{1}$$

in the special case when the right-hand member X is the sum of a number of terms $g_1(x), \cdots, g_j(x)$ such that these terms and all of the terms which arise from them by differentiation can be expressed linearly with constant coefficients by means of a finite number

$$g_1(x), g_2(x), \cdots, g_l(x) \tag{2}$$

of the terms. Such terms for example are x^ρ, $e^{\alpha x}$, $\sin^\rho \beta x$, $\cos^\rho \gamma x$, $\sinh^\rho \delta x$, $\cosh^\rho \epsilon x$ (ρ any positive integer), and any products of these.

As is customary we call $f(m) = 0$ the *auxiliary equation;* and the complete solution of the differential equation $f(D)y = 0$, the *complementary function* of (1). Each r-fold root, $m = \alpha$, of $f(m) = 0$ contributes a part

$$G(\alpha, r) = (\lambda_1 x^{r-1} + \lambda_2 x^{r-2} + \cdots + \lambda_r)e^{\alpha x} \tag{3}$$

to the complementary function.

The rule for the determination of the particular integral U of (1) can be stated as follows:

I. *Substitute for y in* (1) *the trial integral* $U = c_1g_1(x) + c_2g_2(x) + \cdots + c_lg_l(x)$ *and determine the coefficients $c_1, \cdots, c_l$ by equating coefficients of $g_1, \cdots, g_l$. If*

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[1] Stated by him without proof. For a proof consult Heine, "Der Eisenstein'sche Satz über Reihenentwickelungen aller algebraischer Funktionen," *Crelle's Journal*, Vol. 45 (1853), pp. 285–382.

[2] Read before the Md.-Va.-D. C. Section of the Mathematical Association, at Annapolis, December 15, 1917.

however X contains a term $g(x)$ or a term $x^\rho g(x)$ where $g(x)$ already occurs in the complementary function corresponding to an r-fold root of the auxiliary equation, then in making up the trial integral U, the term $x^{\rho+r}g(x)$ and its derivatives are to be used.[1]

This method for determining U has several advantages. It requires only differentiations and the solution of a system of linear equations. It furnishes an integral expressed in terms of functions similar to those that occur in X. It applies to nearly all types of equations that actually occur in practice. It has the further pedagogic advantage that in any particular example the method justifies itself by furnishing the required integral. The following proof of the range and validity of the method is perhaps of interest only to the teacher or more mature student.

We shall first find an explicit form for the right-hand member X of (1), and second prove that the set described in I of linear equations in $c_1, \cdots, c_l$ is consistent.

By hypothesis the linearly independent functions $g_1(x), \cdots, g_l(x)$ derived from the terms in X and their derivatives are such that

$$\begin{array}{ll} & D_x g_1(x) = g_1' = a_{11}g_1 + a_{12}g_2 + \cdots + a_{1l}g_l, \\ (4) & D_x g_2(x) = g_2' = a_{21}g_1 + a_{22}g_2 + \cdots + a_{2l}g_l, \\ & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ & D_x g_l(x) = g_l' = a_{l1}g_1 + a_{l2}g_2 + \cdots + a_{ll}g_l.^2 \end{array}$$

If we set

$$g_i = b_1h_1(x) + b_2h_2(x) + \cdots + b_lh_l(x),$$

then

$$g_i' = b_1h_1' + b_2h_2' + \cdots + b_lh_l';$$

whence *the l functions g and the l derivatives g' are cogredient under linear transformation with constant coefficients.* If then the characteristic equation of the linear transformation (4) has roots $\beta_1, \cdots, \beta_k$ of respective multiplicities $s_1, \cdots, s_k$ $(s_1 + \cdots + s_k = l)$ we can find l linearly independent linear functions $h_1(x), \cdots, h_l(x)$ of $g_1(x), \cdots, g_l(x)$ such that the system of equations (4) takes the form[3]

$$\begin{array}{ll} & h_1' = \beta h_1 + h_2 \qquad\qquad\qquad\qquad , \\ & h_2' = \qquad \beta h_2 + h_3 \qquad\qquad , \\ (5) & \cdots\cdots\cdots\cdots\cdots\cdots \\ & h_{s-1}' = \qquad\qquad \beta h_{s-1} + h_s \ , \\ & h_s' = \qquad\qquad\qquad\qquad \beta h_s, \end{array}$$

where (5) is a sample of that set of equations of the system which corresponds to a root β of multiplicity s of the characteristic equation.

[1] Cohen's *Differential Equations*, p. 112. So far as the writer is aware the method is first set forth there under the name of "the method of undetermined coefficients."

[2] For a similar system of differential equations cf. Riemann-Weber, *Partielle Differentialgleichungen*, p. 138.

[3] Bôcher's *Higher Algebra*, Chap. XXI.

The solutions of (5) satisfy the system

$$
(6)\qquad
\begin{aligned}
&h_s' - \beta h_s &&= 0,\\
&h_{s-1}'' - 2\beta h_{s-1}' + \beta^2 h_{s-1} &&= 0,\\
&h_{s-2}''' - 3\beta h_{s-2}'' + 3\beta^2 h_{s-2}' - \beta_3 h_{s-2} &&= 0,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &&\\
&h_1^{(s)} - \binom{s}{1}\beta h_1^{(s-1)} + \binom{s}{2}\beta^2 h_1^{(s-2)} \cdots + (-1)^s\binom{s}{s}\beta^s h_1 &&= 0.
\end{aligned}
$$

This is evidently true of the first equation; we have only to verify that it is true of the last when the earlier ones are satisfied. By virtue of the numerical relation

$$\binom{s-1}{a-1} + \binom{s-1}{a} = \binom{s}{a}$$

the last equation can be written

$$[h_1^{(s)} - \beta h_1^{(s-1)}] - \binom{s-1}{1}\beta[h_1^{(s-1)} - \beta h_1^{(s-2)}] + \binom{s-1}{2}\beta^2[h_1^{(s-2)} - \beta h_1^{(s-3)}]$$
$$\cdots + (-1)^{s-1}\binom{s-1}{s-1}\beta^{s-1}[h_1' - \beta h_1] = 0.$$

Since $h_1' - \beta h_1 = h_2$ this is satisfied due to the preceding equation in (6). The complete solution of the last equation of (6) is

$$h_1 = e^{\beta x}H(x, s), \qquad H(x, s) = a_1x^{s-1} + a_2x^{s-2} + \cdots + a_s$$

and the values of $h_2, \cdots, h_s$ derived in turn from (6) are

$$h_2 = e^{\beta x}D_xH(x, s), \qquad h_3 = e^{\beta x}D_x{}^2H(x, s), \cdots, h_s = e^{\beta x}D_x{}^{s-1}H(x, s).$$

Thus $h_1, \cdots, h_s$, and therefore the original g's, can be expressed linearly in terms of $x^{s-1}e^{\beta x}, x^{s-2}e^{\beta x}, \cdots, e^{\beta x}$. Hence

II. *The right-hand member X of* (1) *can be expressed as*

$$X = G_1(\beta_1, s_1) + G_2(\beta_2, s_2) + \cdots + G_k(\beta_k, s_k)$$

where the β's are the distinct roots with respective multiplicities s of the characteristic equation of the matrix $|a_{ij}|$ of the system of equations (4) *and the functions G are of the form* (3).

We can obtain a particular integral U, for each part G, of X and by adding the U, get the required integral. If the $X = G(\beta, s)$ we shall take the general case and suppose that β, the s-fold root of the characteristic equation, is also an r-fold root of the auxiliary equation $f(m) = 0$. Then

III. *The method* (4) *furnishes one and only one particular integral of the form*

$$e^{\beta x}[c_1x^{r+s-1} + c_2x^{r+s-2} + \cdots + c_sx^r].$$

For if in $f(D)y$ we set

$$y = e^{\beta x}(b_1x^{t-1} + b_2x^{t-2} + \cdots + b_{t-1}x + b_t)$$

then the coefficients of $e^{\beta x}x^{t-1}$, $e^{\beta x}x^{t-2}$, $\cdots$, $e^{\beta x}$ in the result are respectively

$$
\begin{aligned}
&f(\beta)\cdot b_1,\\
&f(\beta)\cdot b_2 + \binom{t-1}{1} f'(\beta)\cdot b_1,\\
(7)\qquad &f(\beta)\cdot b_3 + \binom{t-2}{1} f'(\beta)\cdot b_2 + \binom{t-1}{2} f''(\beta)\cdot b_1,\\
&f(\beta)\cdot b_4 + \binom{t-3}{1} f'(\beta)\cdot b_3 + \binom{t-2}{2} f''(\beta)\cdot b_2 + \binom{t-1}{3} f'''(\beta)\cdot b_1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

the law of formation being sufficiently obvious. Under the conditions above, for $t = r + s$, we have to set $b_1, \cdots, b_s = c_1, \cdots, c_s$ when $f(\beta) = f'(\beta) = \cdots = f^{(r-1)}(\beta) = 0$ while $f^{(r)}(\beta) \neq 0$ and equate these coefficients in order to those of $G(\beta, s)$ of which the first r are zero and the last s in order are $\lambda_1, \cdots, \lambda_s$ as in (3). Then the first r equations are identically satisfied and the last s equations suffice to determine $c_1, \cdots, c_s$ in terms of $\lambda_1, \cdots, \lambda_s$. In fact since $f^{(r)}(\beta) \neq 0$ the $(r + 1)$-st equation determines c_1 in terms of λ_1; for the same reason the next equation determines c_2 in terms of λ_2 and c_1, etc. Indeed to within a numerical factor the determinant of the last s equations in $c_1, \cdots, c_s$ is $[f^{(r)}(\beta)]^s \neq 0$.

If X is actually expressed in the form given in II the system (7) is immediately available, but as a rule other forms of X are preferable.

RECENT PUBLICATIONS.

WHO WAS THE FIRST INVENTOR OF THE CALCULUS?

The Geometrical Lectures of Isaac Barrow. Translated by J. M. Child, Chicago and London, The Open Court Publishing Co., 1916. xiv + 218 pages.

An English translation of so important a work as Isaac Barrow's *Lectiones geometricæ* will be greatly welcomed. Few American mathematicians have had access to a translation into English by E. Stone, published in 1735; according to a statement made by W. Whewell in the preface to his Latin edition of *The Mathematical Works of Isaac Barrow*, Cambridge, 1860, Stone's translation "was so badly executed that it cannot be of use to any one." Few readers will object to Child's omission of certain parts of Barrow's geometrical lectures, parts which seem to be of little or no interest at the present time. Child's historical introduction and critical notes greatly assist in the deeper comprehension of Barrow's genial work. In fact, Child has aimed to do much more than simply to supply a translation. He has made a searching study of Barrow and has arrived at